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Evaluation of Flat-Earth Approximation Results for Geopotential Missions

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Simplified calculations can approximate the formal uncertainties in estimates of the spherical harmonic coefficients representing the Earth's gravitational potential. The calculations model the Earth locally as a plane, producing errors negligible for wavelengths shorter than the radius of the Earth. Information derived from observations of low-altitude polar orbiting satellites is considered. With some constraints, the final model uncertainties derive from a priori gravitational field information, specific orbital elements, and parameters describing instrumentation characteristics. We demonstrate how to refine the technique to accept inputs from the currently operational Navstar global positioning system (GPS) constellation and how to use information from partial tensor gravitational gradiometers. This approach is beneficial when evaluating prospective satellite geodesy missions because the covariance analyses for various mission scenarios can be made efficiently and expeditiously. We demonstrate the utility of the flat-Earth approach by comparing results with those of more elaborate and time consuming calculations performed for the European Space Agency ARISTOTELES (Applications and Research Involving Space Techniques Observing The Earth's Field From Low Earth Orbiting Satellite) proposed geopotential mapping mission, the NASA Gravity Probe B Relativity mission, and the NASA/Center National d'Etudes Spatiales Topographic Ocean Experiment Satellite (TOPEX)/Poseidon mission. We also show that the flat-Earth approximations are consistent with reductions of the GPS tracking data obtained by TOPEX/Poseidon.

Nomenclature

e = natural logarithm base
$F(\omega)$ = factor of reduction in geopotential variance at spatial frequency ω
$\boldsymbol{H}(\varpi)$ = information vector associated with instrument observations
h = satellite altitude
j = square root of -1
$K(l)$ = Kaula's rule of thumb for Earth spherical harmonic coefficient magnitudes
l = spherical harmonic degree, approximated ωR_e
R_e = Earth radius, 6.378137×10^6 m
$\boldsymbol{S}$ = space-domain position observation, components S_x, S_y, S_z
$\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ = information in frequency domain associated with position perturbations in the N–S, E–W, and vertical directions
t = time
$U(x, y, z)$ = gravitational potential in topocentric frame in space domain
U_s = potential at the surface of the Earth
$\hat{U}(\omega, z)$ = gravitational potential in frequency domain; ω is scalar
V = satellite velocity, $\mathrm{d}x/\mathrm{d}t$, $[\mu_e/(h + R_e)]^{1/2}$
x = northward (along-track) space-domain coordinate
y = westward (cross-track) space-domain coordinate
z, z_1, z_2 = upward coordinates in frequency and space domains
α, ω = polar coordinates of frequency-domain vector (ω_x, ω_y)
$\hat{\Gamma}_{ij}$ = component of gravitational gradient tensor, $i, j \in \{x, y, z\}$
Γ_{ij} = frequency-domain information associated with Γ_{ij}
μ_e = gravitational constant times Earth mass, 3.986×10^{14} m^3/s^2
Φ_W^{-1} = diagonal matrix of white noise associated with observations
$\phi_{U_i}(\omega)$ = a priori uncertainty in geopotential at frequency ω

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I. Introduction

THE flat-Earth method of Breakwell[1] approximates the formal uncertainty in an estimate of the spherical harmonics describing the geopotential. The uncertainty is expressed as a function of the wavelength of undulations in the geopotential field at the Earth's surface. The wavelength approach does not consider magnitudes of the spherical harmonic coefficients nor does it distinguish among specific coefficients at the same wavelength. The user may choose any combination of orbital altitude, instrumentation, and mission duration but is restricted to a circular polar orbit. The flat-Earth approximation requires that the separation between observations be less than the shortest surface geopotential wavelength considered and that the density of observations be uniform.

Breakwell[1] developed equations representing the information content of different types of satellite data. He included high–low satellite observations of relative velocity, but did not consider precise geometrical positioning of satellites. The Navstar global positioning system (GPS),[2] commissioned on March 9, 1994, permits precise geometric position measurements in a high–low satellite configuration with continuous global coverage. Breakwell[1] also examined the utility of partial tensor gravitation gradiometry, but presented calculations only for its largest component: vertical strain in a geocentric frame.[3] (In this paper gravity refers to force felt in an Earth-fixed rotating frame, whereas gravitation refers to force in an inertial frame.) A recent development is the proposal to apply the use of highly precise specific force meters to orbiting partial-tensor gradiometers observing other than the vertical strain component.[4] The use of partial-tensor gradiometers, but not the general treatment, was discussed by Schaechter et al.[5]

We extend Breakwell's technique by developing equations representing information associated with position observations of a single satellite to permit GPS information to be analyzed. We also present an ad hoc correction to Breakwell's treatment of geopotential estimate uncertainty for the case where only data from a horizontal strain single component gradiometer are available. With this modification, the equations of Breakwell representing information from partial-tensor gradiometers [see paragraph introducing Eq. (9)] do not produce anomalously large approximations to the uncertainty for most conditions. These two refinements make the simple flat-Earth approximation much more powerful. We compare the results from our equations with more elaborate independent calculations for the ARISTOTELES (Applications and Research Involving Space Techniques Observing The Earth's Field From Low Earth Orbiting Satellite),[6] Gravity Probe B Relativity mission (GP-B), and Topographic Ocean Experiment Satellite (TOPEX)/Poseidon[7] satellites. We also compare them with reductions of TOPEX/Poseidon GPS observations (i.e., experimental data). We show that the simple yet flexible flat-Earth approximation yields results compatible with the other methods within its boundary conditions.

II. Background

The flat-Earth method studies the observability of Earth's geopotential harmonic coefficients by satellite instruments rather than the instrumental sensitivity to variations of the potential field. Its purpose is to incorporate the information associated with a set of new observations together with the estimated error in an a priori estimate of the geopotential to approximate the formal error associated with an improved estimate. It is a covariance analysis that addresses neither the actual observations nor the coefficients describing the geopotential, but the uncertainties associated with both.

The flat-Earth method[1] employs a two-dimensional Fourier transformation to consider the upward extension of the gravitational field in a hybrid domain consisting of a plane normal to the radius with two frequency coordinates, and a spatial radial coordinate. The resulting convolution represents attenuation of geopotential signals with increasing radius in a spherical field. A truncation of the convolution produces a simple expression, which can be used to model observability of the geopotential to various satellite instruments. The tedious double integral of the Laplace equation is replaced with a simple multiplication but the characteristics of a spherical field with increasing radius are retained.

Directional content is sacrificed to reduce the use of complex numbers; further, the constants of integration representing zero-frequency terms are omitted. The flat-Earth method approximates only a formal uncertainty as a function of wavelength associated with a geopotential estimate. Instrument coverage is assumed to be globally uniform, which normally rules out ground-based tracking systems. The longest wavelengths for which the truncation produces acceptable errors are those somewhat shorter than the radius of the Earth, i.e., spherical harmonic degree greater than 6. The coefficients for the lower degrees and orders have been established to approximately a part in 10^7 (degree 2 and order 0) or a part in 10^4 from topocentric observations of the LAGEOS (Laser Geodesy Satellite) and are not usually at issue.

The result of a flat-Earth calculation for a given combination of instrument and mission parameters and a priori geopotential uncertainty is a reduction factor $F(\omega)$ between 0 and 1 for each wavelength of interest:

$$F(\omega)^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\alpha}{1 + \phi_{U_i}(\omega)\boldsymbol{H}^T(\varpi)\Phi_W^{-1}\boldsymbol{H}(-\varpi)} \tag{1}$$

The product of this factor and the a priori uncertainty approximates the uncertainty of the geopotential field estimate from the satellite system described by the input parameters.

Section III considers the derivation of formulas allowing the inclusion of information associated with data from instruments sensing position. In the flat-Earth approximation, these formulas represent transfer functions between surface geopotential perturbations and satellite observations. Section V deals with a special case of the transfer function with only limited gravitational gradient information.

Sections IV and VI present graphs giving results for a range of wavelengths. For the ordinates, we normalize wavelength by dividing by R_e to represent spherical harmonic degree l. For the abscissas, we have similarly used the variance associated with a given wavelength interchangeably with the rms of coefficients of all orders at the degree corresponding to that wavelength. However, we recognize that the spectral response of a satellite instrument to a specific spherical harmonic depends not only on the degree of the spherical harmonic, but also the order and on the orbital elements of the satellite.[8] Any cavalier interchange of degree, order, and wavelength implied by the graphs should be treated with caution. In keeping with common practice, we at times also utilize Kaula's[8] rule of thumb, which is intended to give coefficient magnitudes, as an a priori for the uncertainty associated with the coefficients.

III. Position Data Type Extension

Whereas satellite ephemerides are conventionally expressed in a geocentric spherical reference frame, the operational frame for the flat-Earth approximation is a set of parallel tangential planes separated by radial increments. We start to derive equations representing the information content of position observations by calculating a position S in Breakwell's[1] orthogonal right-handed topocentric coordinate frame,

$$\{S_x, S_y, S_z\} = \iint \left\{\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z},\right\} \mathrm{d}t\,\mathrm{d}t \tag{2}$$

We express the vector $\boldsymbol{S}$ as three double integrals with respect to time of the acceleration associated with perturbations of the geopotential U at the satellite. We discard the constants of integration because we are considering sensitivity rather than actual position. Similarly, the formal uncertainties of the data are of interest rather than the position observations themselves. The braces { } notation implies a choice (which remains consistent throughout the equation) of one of its contents; it does not indicate a vector or tensor quantity.

Introducing the factor $V = \mathrm{d}x/\mathrm{d}t$ changes the independent variable to x,

$$\{S_x, S_y, S_z\} = \frac{1}{V^2}\iint \left\{\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z},\right\} \mathrm{d}x\,\mathrm{d}x \tag{3}$$

The x and y coordinates will be transformed from the space domain to the frequency-domain coordinates ω_x and ω_y in the three-dimensional hybrid domain used by the flat-Earth approximation. As in Breakwell,[1]

$$\hat{U}(\omega, z) = \iint \exp[-j(\omega_x x + \omega_y y)]U(x, y, z)\,\mathrm{d}x\,\mathrm{d}y \tag{4}$$

Breakwell[1] gives the flat-Earth expression in the frequency domain for the upward attenuation of the strength of Earth's gravitational potential U between altitudes z_1 and z_2 at frequency ω. It is analogous to the effect found in solutions to Laplace's equation, which meet the condition of finite potential at infinite radius,

$$\hat{U}(\omega, z_2) = \exp[-(z_2 - z_1)\omega]\hat{U}(\omega, z_1) \tag{5}$$

We combine Eqs. (3) and (5) for the potential, set $z_1 = 0$ (the surface in a topocentric frame) and $z_2 = z =$ satellite altitude and expand $\hat{U}(\omega, z)$ using Eq. (4) to obtain

$$\{\hat{S}_x, \hat{S}_y, \hat{S}_z\} = \frac{1}{v^2}\iint \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} e^{-z\omega} \times \iint \exp[-j(\omega_x x + \omega_y y)]U(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}x \tag{6}$$

$\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ represent the information content in the frequency domain associated with geopotential-induced position perturbations in the north–south, east–west, and vertical directions, respectively. Performing the partial differentiation with respect to x, y, or z reduces the expression to the position information content in the frequency domain; the derivatives are applied to the terms describing frequency content, i.e., the exponential terms. $U(x, y, z)$ is treated as locally constant. This produces an expression for the potential multiplied by the factors $-j\omega_x$, $-j\omega_y$, and $-\omega$. We next perform the integrations with respect to x, which modify the integrand by a factor of $(-j\omega_x)^{-2}$. We express the result as a transfer function from the potential at the assumed reference surface of the flat Earth

($\hat{U}_s$) representing the local geopotential in the neighborhood of the subsatellite point to position observations at the satellite:

$$\frac{\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}}{\hat{U}_s} = e^{-z\omega}\left\{\frac{-1}{V^2 j\omega_x}, \frac{j\omega_y}{(V\omega_x)^2}, \frac{-\omega}{(V\omega_x)^2}\right\} \tag{7}$$

The three right-hand expressions give components of the $\boldsymbol{H}(\varpi)$ vector in Eq. (1), which represent the transfer functions from the surface geopotential to along-track, cross-track, and vertical position observations, respectively.

Equation (7) represents the information content of three-dimensional observations of the satellite's position in a topocentric reference frame. With these expressions, the flat-Earth method can use information associated with position observations to approximate the expected improvement in the formal uncertainty of the geopotential coefficients from the ephemerides of low-altitude satellite missions that meet the other conditions of the flat-Earth approximation.

Some GPS receivers employ interferometric techniques using the received carrier phase rather than its modulation to obtain relative positions only, with precisions (approximately 1 cm) much better than one carrier wavelength (20 cm). The interferometer utilizes the cumulative phase shift of the received carrier frequency and, thus, the Doppler shifted transmitted frequency. It is not recommended that Doppler observations and interferometry be combined. Calculations using both relative velocity and relative position information from a phase-sensitive receiver will produce overly optimistic results. For purposes of the flat-Earth approximation, we treat phase-sensitive GPS receivers as a source of relative position data only. Velocity would be determined from position and time information if desired.

GPS positioning for satellites is subject to some physical limitations, any of which may produce systematic errors in the calculated position, but the flat-Earth method considers only white noise in the position measurement. Therefore, the value chosen for the observation noise must be great enough to include systematic errors over the spatial wavelength range of interest.

There is one other caveat about the use of the flat-Earth method for GPS-equipped satellites. Satellite surface forces change the ephemeris from a purely gravitational one, diluting the quality of the estimated geopotential. A proof mass drag-free sensor in GP-B allows its control system to isolate its trajectory from nongravitational influences. Except for such drag-free satellites, the flat-Earth results can represent only a lower bound on uncertainty.

IV. Position Data Type Calculations

There are techniques ranging from the simple to the complex for estimating the precision of results from various satellite geodesy experimental proposals. The flat-Earth assumptions represent the simple and direct procedure. We wish to demonstrate that its approximations, within its boundary conditions, are comparable to the more elaborate treatments. We compare uncertainty approximations from our position information equations to those of other methods for representative test cases. We choose two satellites whose contributions have been evaluated by independent methods, which meet the boundary conditions of the flat-Earth method and which have been or are being considered for future missions: the European Space Agency's ARISTOTELES and NASA's GP-B.

ARISTOTELES was a combined gravitation and magnetic field observing mission, which would have required a variety of different orbits. ARISTOTELES was to be neither drag free nor polar. Its primary geodetic instrument was to be a gravitation gradiometer of 0.01 Eötvös unit sensitivity; see Sec. IV for analysis of this instrument's observations. In the study by Visser et al.,[6] ARISTOTELES was assumed to be in a 200-km sun-synchronous circular orbit, and 3 months of data (one repeat period) with one observation per minute were represented.

Approximations of the quality of the geopotential estimates from GPS inputs using the ARISTOTELES position data set were made by Visser et al.[6] using a method based on Kaula's[8] frequency analysis technique. Visser et al. studied four cases. Case 1 assumes a pseudorange only GPS receiver, which recovers a position with a precision of 3 m in all three axes. Case 2 was a tabulation of the formal uncertainties of the Goddard Earth Model for TOPEX, version 2 (GEM-T2) Earth geopotential model.[9] Case 3 uses the GEM-T2 uncertainties as a priori information with the GPS receiver of case 1. Case 4 assumes a phase-sensitive GPS receiver with a resulting position observation precision of 10 cm.

Neither the flat-Earth nor the Visser et al. method[6] considers the effects of systematic unmodeled errors and nongravitational forces. Both assume that phase ambiguity is resolved so that the GPS measurements contain systematic errors of only the order of their precision. The approximation to the formal uncertainty in the geopotential is displayed as an average (over all orders at a given degree) uncertainty in spherical harmonic coefficients as a function of degree. Except as noted subsequently, the a priori information for the flat-Earth approximation was taken to be an average coefficient uncertainty of 10^{-6}, large enough to have no significant effect on the results shown here.

The plots we present include the anticipated average magnitude of the geopotential harmonic coefficients derived from Kaula's rule.[8] The formula is normally

$$K(l) = \frac{10^{-5}}{\sqrt{2}(\omega R_e)^2} = \frac{10^{-5}}{\sqrt{2}(l)^2} \tag{8}$$

The combination ωR_e is taken to be equivalent to the degree l of spherical harmonic coefficient. The factor $\sqrt{2}$, sometimes used with Kaula's[9] formulation, is omitted in Fig. 1a to agree with Visser et al.[6]

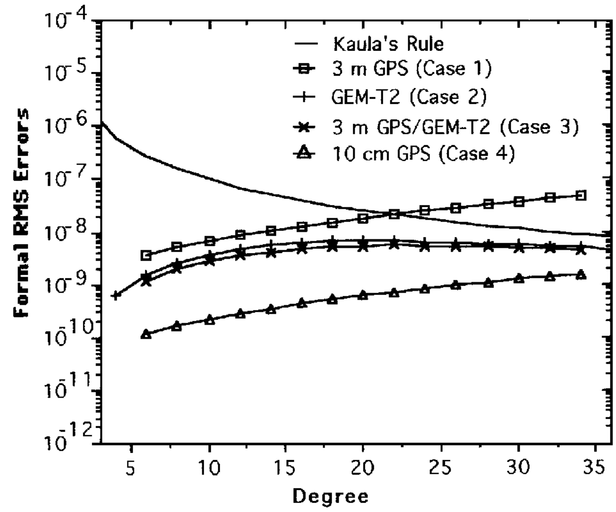

Fig. 1a Flat-Earth approximations to geopotential estimate uncertainty using ARISTOTELES GPS data for circular polar 200-km orbit; 3 months of observations at 1 per minute.

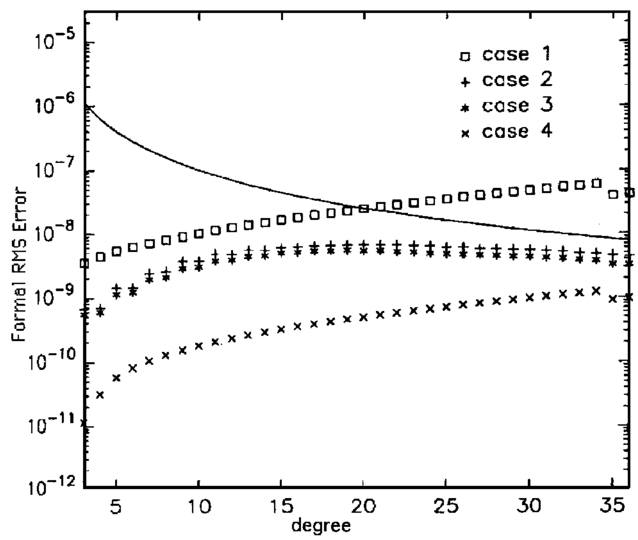

Fig. 1b Formal rms errors of low-degree spherical harmonic coefficients as calculated by Fig. 4 of Visser et al.[6]

Kaula derived his formula for coefficient magnitudes from surface observations, and it is therefore not expected to be representative at low degrees and orders.

Figure 1a shows the flat-Earth results for the same four cases of Visser et al.,[6] and Fig. 1b is a reproduction of the Ref. 6 figure. The Visser et al. and Breakwell's methods are both predictions for a mission that has not taken place as of this writing and, therefore, neither has been substantiated by analysis of actual data, but agreement between them shows that for this case there is no advantage to using the more computationally intense technique of Visser et al. The flat-Earth method easily accommodates their boundary conditions once the $\boldsymbol{H}(\varpi)$ vector entries for position information are available.

In case 3, both the flat-Earth and the Visser et al. method show that the a priori information dominates, so that agreement between them is not quantitative. We omit the flat-Earth results at degrees less than 6, in keeping with our statement that it should not be expected to produce reliable results at wavelengths longer than the Earth's radius. As we hoped to show, the quantitative results of the two methods are similar above degree 6. Both depart from Kaula's rule at the low degrees; we use the rule to permit quantitative comparisons among the various techniques.

The most elaborate technique to evaluate satellite geodesy experiments is simulation of both the data collection and reduction process. This method produces estimates and variances for each coefficient. The covariance matrix associated with reduction of simulated data indicates the formal uncertainty that will be associated with the geopotential estimate reduced from the final data set. We next match the flat-Earth method against a simulation.

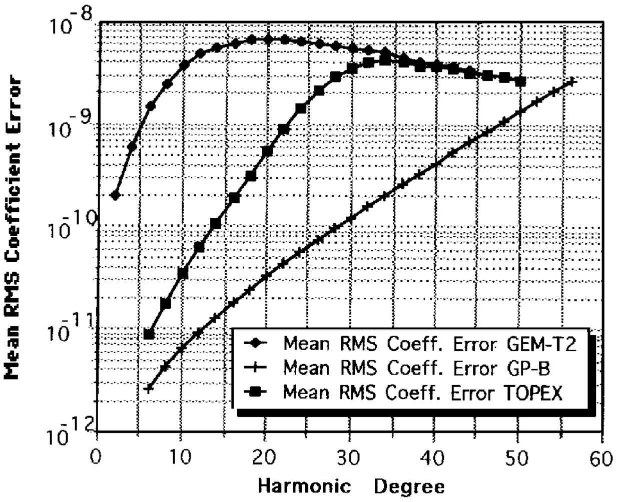

Fig. 2a Flat-Earth approximation to formal uncertainty associated with geopotential estimate from GP-B and TOPEX/Poseidon GPS data with phase ambiguity resolved.

To investigate how the flat-Earth compares to simulations, we use calculations by Pavlis[7] for the NASA GP-B and NASA/Center National d'Etudes Spatiales TOPEX/Poseidon mission (described subsequently). Pavlis utilized the sophisticated software used to reduce satellite data for the GEM models to simulate and reduce sets of data for GP-B and TOPEX/Poseidon. For GP-B, he assumed a 600-km circular polar orbit for 24 months. For TOPEX/Poseidon, he assumed a 1336-km circular orbit of inclination 66 deg for 6 months. Both simulations assumed GPS receivers with phase observations of 2-cm precision collected once per second. To reduce computations Pavlis used normal points representing combined results of 40 and 60 observations for GP-B and TOPEX/Poseidon, respectively. Pavlis considered that the phase ambiguity was always resolved. (Colombo[10] showed that unresolved phase ambiguity was significant only below degree 8.) TOPEX/Poseidon's orbit does not cover the polar regions, and so estimating a complete geopotential model from only TOPEX/Poseidon data is an ill-posed mathematical problem. Therefore, the TOPEX/Poseidon simulated data were combined with GEM-T2 normals. In both cases only a 10-day arc was simulated, with the error associated with each normal point divided by the appropriate factors (73 for GP-B and 18 for TOPEX/Poseidon) to reflect error averaging for observations over missions of 730 and 180 days, respectively.

Figure 2a shows flat-Earth results with the same parameters as Pavlis[7] for mission length and altitude. Data rates were adjusted to one three-axis set of position observations (x, y, and z) of accuracy 4 and 3 mm every 40 and 60 s for 24 months at 600-km altitude and 6 months at 1336-km altitude for GP-B and TOPEX/Poseidon, respectively. Position observations are used to represent the phase observations and, implicitly, the modeling of the GPS orbits and other aspects of the observation system of Pavlis' study. (This does not imply that the orbits are known to the 3–4 mm level.) The degree rms coefficient uncertainty from GEM-T2 is used as the a priori for the TOPEX/Poseidon calculation, while a constant of 10^{-6} is used as the a priori for GP-B. Figure 2a includes average covariances from the GEM-T2 model for comparison, as does Pavlis. Figure 2b is Pavlis'[7] Fig. 2 showing for GP-B 2 years at 600-km altitude, 40-s normal points of accuracy 4 mm and for TOPEX 6 months at 1336-km altitude, 60-s normal points of accuracy 3 mm. All phase ambiguities are assumed resolved.

The flat-Earth results have a similar shape to Pavlis'. Pavlis' plot of GEM-T2 includes every degree, whereas the flat-Earth plot only gives even degrees; the odd degrees produce a bit of jaggedness in the curve at low degrees. Pavlis' calculations use the full normals; in this case the correlations or covariances are considered, not just the

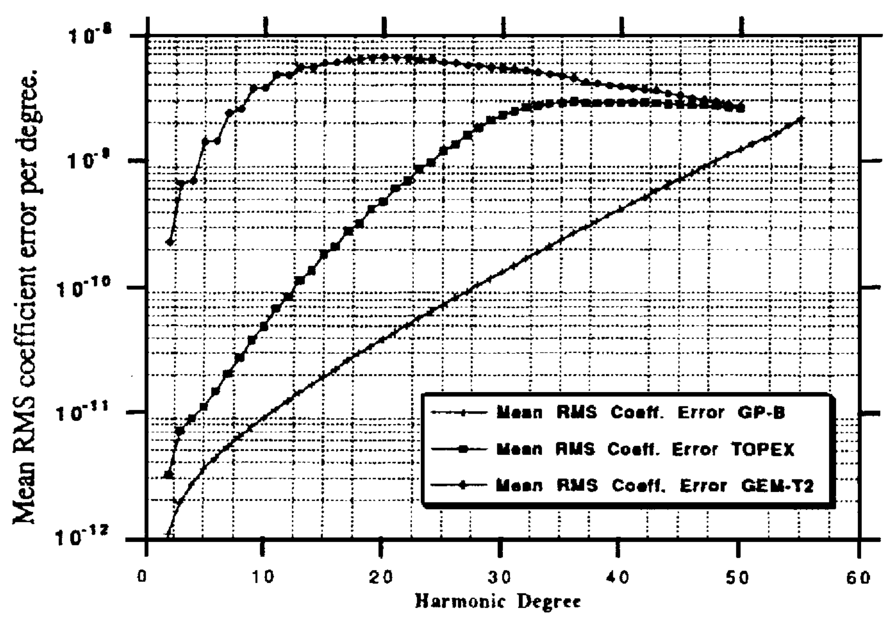

Fig. 2b Gravitational spectra from GPS tracking of future missions: GP-B and TOPEX/Poseidon from Pavlis' Fig. 2.

variances. The flat-Earth TOPEX/Poseidon curve, therefore, converges to the GEM-T2 curve more gradually than Pavlis'. The trends are comparable so that both techniques are useful. As expected, agreement is better for GP-B, the polar orbit of which meets the flat-Earth method's boundary conditions. Even when those conditions are severely relaxed (as with TOPEX/Poseidon's inclination) the approximation does not appear to be substantially degraded. There is a significant difference in computation time. Pavlis invested 8 h and 24 min on a Cyber 205 considering all orders for each degree. The flat-Earth calculation required about 8 s on a Macintosh Centris 610.

Finally, we compare the flat-Earth method with data reduced from observations. A currently available geopotential estimate, which contains such data, is JGM-3 (NASA/University of Texas Center for Space Research Joint Gravity Model 3).[11] JGM-3 was developed from JGM-1 by including 1,239,551 GPS position observations from the TOPEX/Poseidon satellite along with a smaller amount of satellite laser ranging data. The TOPEX/Poseidon GPS and altimetry data contained in the JGM-3 solution contributed to the improvement in the variances for the higher degree coefficients. The jointly sponsored U.S./France TOPEX/Poseidon satellite, launched on Aug. 10, 1992, is a radar altimetry experiment in a near-circular (eccentricity 0.0001) orbit of 1336-km altitude at an inclination of 66.0 deg (Ref. 12) for studying the ocean surface.

The JGM-3 estimate included all of the information from JGM-1, using data from many satellites at varying altitudes and inclinations. For comparison, we approximate improvement from JGM-1 to JGM-3 resulting from the addition of TOPEX/Poseidon GPS information. Our objective is to show that the flat-Earth approximation of estimate uncertainties is comparable to the uncertainties associated with the JGM-3 estimate. We display flat-Earth and JGM results on the same plot. We use the variances associated with JGM-2 as the a priori for the flat-Earth method. The JGM-2 variances are very near to those of JGM-1[13] and are more readily available. We assume 1,239,551 three-axis GPS topocentric position observations.

The results by Visser et al.[6] and Pavlis[7] presented earlier assumed GPS phase observations with 1–2 cm precision. To compare flat-Earth results with theirs, we made similar assumptions. However, real-world data reductions display increased residuals caused by perturbations due to radiation pressure or drag (Pavlis' calculations[7] assumed none), model errors, or model omissions. Higher residuals may result in a decision to downweight the experimental observations to make the formal variances of the estimate consistent with the residuals.

A precision of 1 cm represents a lower limit for the uncertainty associated with these GPS phase observations, and we present calculations based on this value. As a second case, we select an uncertainty of 20 cm. Given the impossibility of predicting the real-world weight, this arbitrary uncertainty may serve a useful purpose.

Figure 3 shows the JGM-2 variances which were a priori, the 1-cm and 20-cm flat-Earth cases, and the JGM-3 variances. For reference, typical values of the coefficients themselves vary between 0.9×10^{-7} at degree 10 and 0.2×10^{-8} at degree 40 (Ref. 11). The flat-Earth program was modified to use the JGM-2 averaged variances as a priori. The boundary conditions describing orbit altitude, duration, etc., were input as usual.

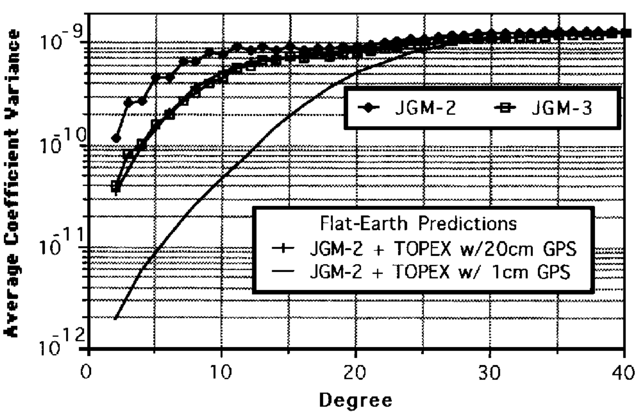

Fig. 3 Comparison of flat-Earth approximation to reported variances for JGM-3 with JGM-2 a priori and TOPEX GPS data.

Figure 3 shows that the 1-cm flat-Earth results are more optimistic than the JGM-3 variances. The flat-Earth case with GPS precision degraded to 20 cm matches the JGM-3 variances very closely. That reduction in precision is consistent with data weighting procedures (downweighting) in JGM-3. The weighting of data to about one GPS carrier wavelength is probably coincidental. Although no technique can accurately predict the final weight that will be assigned to a data set, theoretical techniques are useful in showing trends, giving uncertainty resulting from various possible data weights, and providing comparative analysis of missions with different parameters. Figure 3 shows that the flat-Earth method can produce results consistent with the JGM-3 estimation process.

V. Gradiometer Data Type

A gravitation gradiometer senses the gravitational gradient by comparing specific force at two separate locations. The commonly used unit of gravitation gradient is the Eötvös, or $10^{-9}/\mathrm{s}^2$ (acceleration per unit length). Here we deal with instruments that measure only along-track or cross-track strain components of the gravitation gradient.

The gravitational gradient tensor can be represented in an orthogonal coordinate system using the gravitational potential derivatives along two axes using all axis permutations.[3] This tensor of nine symmetric components has trace zero. Here we use the same topocentric reference frames as before. The flat-Earth gives expressions that relate information in the hybrid transformed frame associated with the Fourier transforms of observations in the topocentric frame. The expressions for the nine elements of the gravitation gradient tensor [again given in the form of entries to the $\boldsymbol{H}(\varpi)$ vector] are[1]

$$\left\{\begin{matrix} \hat{\Gamma}_{xx} & \hat{\Gamma}_{xy} & \hat{\Gamma}_{xz} \\ \hat{\Gamma}_{yx} & \hat{\Gamma}_{yy} & \hat{\Gamma}_{yz} \\ \hat{\Gamma}_{zx} & \hat{\Gamma}_{zy} & \hat{\Gamma}_{zz} \end{matrix}\right\} \frac{1}{U_s} = e^{-\omega z} \left\{\begin{matrix} -\omega_x^2 & -\omega_x\omega_y & -j\omega_x\omega \\ -\omega_x\omega_y & -\omega_y^2 & -j\omega_y\omega \\ -j\omega_x\omega & -j\omega_y\omega & \omega^2 \end{matrix}\right\} \tag{9}$$

The tabulated form of Eq. (9) puts each information equation in the position of the gradient tensor element with which it is associated, but the tensor properties of the gradient are not preserved. We again use the braces notation to indicate choice.

The relations in Eq. (9) are used to construct the $\boldsymbol{H}(\varpi)$ vector of Eq. (1). Equation (1) was arranged for computational efficiency, but multiplying numerator and denominator of the integrand by $\phi_{U_i}^{-1}(\omega)$ yields a more transparent form for inspection. The numerator and the first addend in the denominator of Eq. (1) represent the a priori information. The second addend in the denominator represents information from the new observations. A value for Φ_W, which is large compared to $\phi_{U_i}(\omega)$, causes $F(\omega)^2$ to approach unity; that is, such conditions allow very little improvement in uncertainty. Recall that the noise factor matrix Φ_W contains effects of measurement precision and averaging of repeated measurements as well as noise.

The $\boldsymbol{H}(\varpi)$ information expression vector for a gradiometer consists of a term chosen from the right-hand side of Eq. (9) for each gradient tensor component to which the instrument is sensitive. Some terms in Eq. (9) (e.g., $\omega_x = \omega\cos\alpha$) depend on α, and so the $\boldsymbol{H}(\varpi)$ vector for an instrument can be zero for some values of α if the instrument observes a combination of components of ω_x or ω_y only. Depending on the a priori and observation strength, the overall integrand magnitude may change dramatically for those values of α.

The zeros reflect a prediction by the flat-Earth approximation that some specific coefficients at a given wavelength are not observable by that instrument for the gradient tensor component considered. Gradiometer instruments that are sensitive to a small subset of the full tensor, e.g., cross-track strain only, are the only ones that produce this outcome. The effect would be less pronounced for a spherical Earth calculation, because such real-world effects as convergence of ground tracks at the poles and ground track crossover due to Earth rotation, which can have significant benefits for horizontal gradiometers, must be neglected in flat-Earth calculations. In comparisons, flat-Earth approximations for gradiometer systems without trigonometric zeros agree with more sophisticated spherical Earth models, but the uncertainties are significantly larger for gradiometers with zeros. A realistic refinement would remove the effects of $\boldsymbol{H}(\varpi)$ vector zeros from the flat-Earth calculations with the understanding that the resulting prediction may not apply to all coefficients in a spherical-Earth estimate.

The flat-Earth results can be made to agree with the others by excising a portion of the definite integral near the zero. Figure 4a illustrates the results of the procedure with an empirically determined excision of $2\pi/30$ rad of the total 2π-rad integral in ranges centered on the two zeros. The size of the excision was chosen to give optimum results for this case.

VI. Gradiometer Calculation Comparisons

We compare the flat-Earth results against another analysis of the ARISTOTELES mission. Schrama[14] published approximations of the uncertainty of a geopotential estimate using gradient information from the components of the tensor one at a time. His method of analysis is similar to that of Visser et al.,[6] but uses a different portion of the ARISTOTELES mission. He assumed an altitude of 200 km, eccentricity 0.0001, inclination 90 deg, and duration 6 months (the full duration of the low-altitude phase of the mission, rather than one repeat cycle; see also Ref. 6). A single-component gradiometer of precision 0.01 Eötvös collected an observation every 4 s. We analyze two of the different instrument configurations Schrama[14] considered: Γ_{zz} (vertical strain) and Γ_{xx} (along-track strain) observations. The flat-Earth approximation does not distinguish between Γ_{yy} and Γ_{xx} observations alone. Γ_{zz} is not a function of α and, hence, cannot contain trigonometric zeros, but Γ_{xx} can. We present the flat-Earth approximation for these conditions in Fig. 4a and Schrama's comparable results in Fig. 4b. (Schrama's figure also plots four other components, which Fig. 4a omits for clarity.) Schrama's nomenclature was changed in Fig. 4b to agree with ours. Typical values of the coefficients themselves vary from 0.8×10^{-7} at degree 10 to 0.1×10^{-8} at degree 70 (Ref. 11). The value of the degree-2, order-0 term is known to about 0.5×10^{-10} (a part in 10^7). Both techniques confirm that vertical strain observations produce estimates with the lowest formal uncertainties for a given instrument precision.

The efficient flat-Earth method produces results that agree well with Schrama's treatment between degree 10 and 70. The disagreement at low degrees is to be expected in light of the constraints mentioned earlier with respect to the flat-Earth approximation.

To preserve the less complex flat-Earth approach, the excision technique should be considered should zeros of the $\boldsymbol{H}(\varpi)$ vector occur. An excision of the integral produced reasonable agreement with the more sophisticated analysis in the case of ARISTOTELES. Excision need be considered only when all elements of the $\boldsymbol{H}(\varpi)$ vector have zeros for the same value of α; the addition of either a vertical strain gravitational gradient observation, a second horizontal strain observation in a perpendicular axis, or position measurements (see Sec. III) for ARISTOTELES would obviate the need for it.

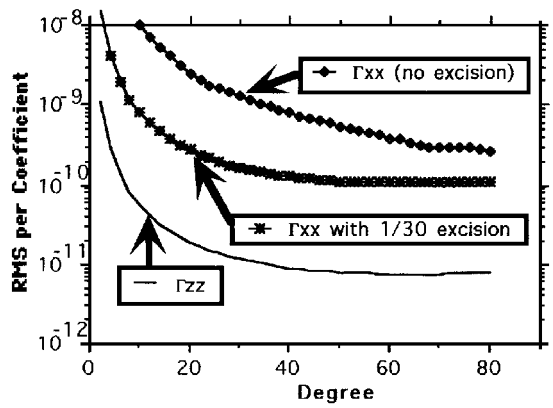

Fig. 4a Flat-Earth approximation of ARISTOTELES gradiometer determination of geopotential field with and without compensation for trigonometric zeros in Γ_{xx}.

VII. Summation

The spectrum of methods to evaluate satellite geodesy missions extends from the simplicity of the flat-Earth technique through methods of Visser et al.[6] and Schrama[14] to the highly complex simulations by Pavlis.[7] Figure 5, a summary of relevant results from Figs. 1–4, shows that results from the flat-Earth method are comparable to those from other techniques. The trends are very similar, except at degrees so low that the flat-Earth approximation is not applicable. At high degrees, all methods converge as the magnitude of the coefficients diminishes.

All of the analysis techniques we compared identify precision for proposed experiments. Appropriate weighting of experimental data cannot be resolved prior to availability of the actual residuals, and perhaps not even then. Nevertheless, for mission design and comparison purposes, predicting trends, and calculating a possible range of outcomes, any of the techniques can serve. All else being equal the simple approach should suffice. Computation times vary from 8 s on a personal computer to about 8.5 h on a Cyber 205. The ability of the flat-Earth technique to accommodate missions employing a variety of sensors including position observations from GPS and/or full or partial tensor gravitational gradient observations adds to its flexibility and utility.

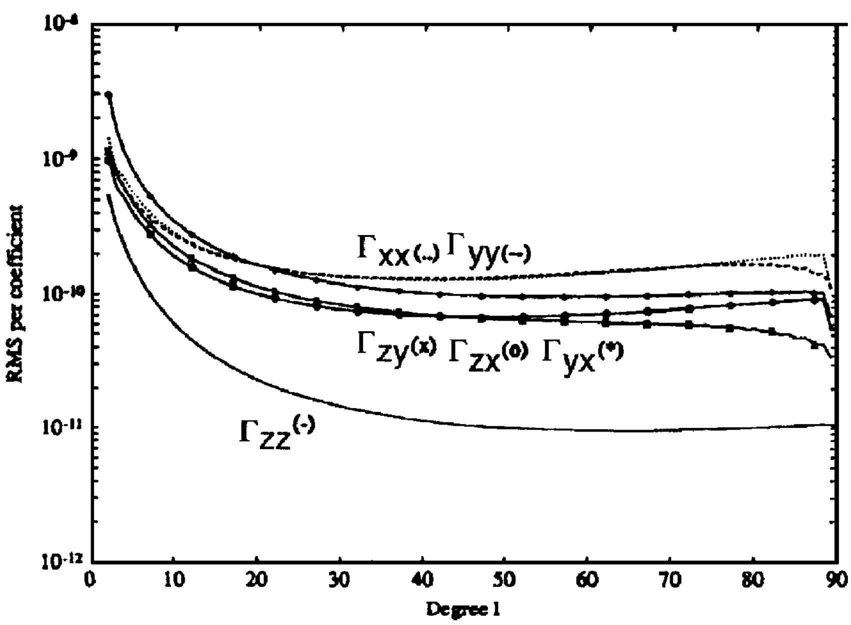

Fig. 4b Results comparable to Fig. 4a extracted from Ref. 14.

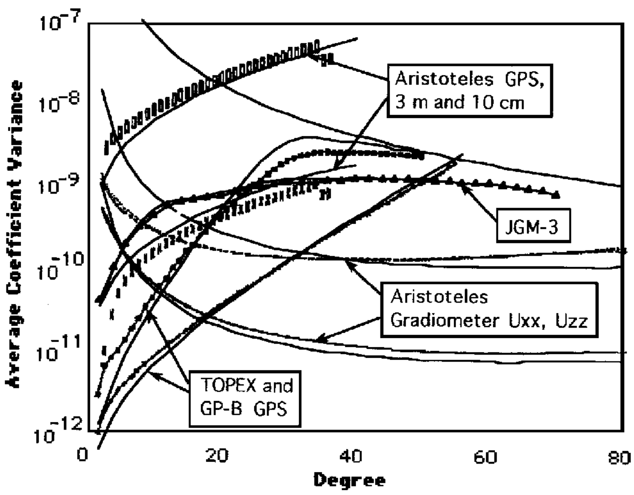

Fig. 5 Combined applicable results of Figs. 1–4 showing wide range of applicability of flat-Earth approximation: flat-Earth results (and Kaula's rule[9] including the square root of 2 divisor) shown by thin lines; results from other authors marked by symbols.

VIII. Conclusion

The flat-Earth method approximates the formal uncertainty in geopotential estimates based on information content of data collected by satellite-borne systems. Comparative analyses of the effectiveness of various orbit altitudes with different combinations of gradient tensor component observations and/or other instrument types can be performed (once the basic program is set up) more expeditiously than by most other techniques. The flat-Earth method has an efficiency advantage because it addresses only frequency content rather than treating each spherical harmonic individually. The condition that the orbit be polar and circular is well aligned with requirements for a global geodetic satellite mission, yet the flat-Earth method can produce acceptable approximations for position observations of orbits with inclinations as low as that of TOPEX/Poseidon (66 deg). The refinements we suggest allow the flat-Earth method to deal conveniently with the powerful and globally available GPS geometrical observations and with gradiometry of any single component of the full gravitational gradient tensor. We demonstrated that flat-Earth results are comparable with more sophisticated estimates produced by three different methods. Comparison to a geopotential field model (JGM-3) derived from TOPEX/Poseidon GPS data with JGM-1 as a priori showed that flat-Earth approximations can be consistent with formal uncertainties associated with reductions of data from satellites.

Timely evaluations of the formal uncertainty in geopotential estimates for anticipated satellite data sets are very useful in establishing priorities among satellite geodesy experimental proposals. Screening of mission proposals using the flat-Earth method would be adequate and demand fewer computer resources than the more elaborate alternatives. Because the flat-Earth approximation produces adequate approximations at a small fraction of the cost and time of other techniques and because its boundary conditions most likely include the majority of satellite geodetic missions, it is the appropriate tool for preliminary analyses of potential satellite system proposals.

Appendix: Summary of Necessary Results from Breakwell

The principal result of the flat-Earth method is a formula which defines a scalar average reduction factor $F(\omega)$ of an a priori uncertainty (variance) of the Earth's geopotential at a spatial frequency ω. The computational procedure is a numerical quadrature at each wavelength of interest. No large matrix manipulation is required, and so the calculations may be carried out with small computational resources. Equation (1) is Breakwell's[1] Eq. (19) for $F(\omega)$ adopting our notation. The definite integral in the frequency domain runs from 0 to 2π rad in the variable α.

The a priori geopotential uncertainty factor $\phi_{U_i}(\omega)$ is the product 4π times the square of the geopotential uncertainty times the square of μ_e.

The diagonal elements of Φ_W^{-1} were calculated by dividing the number of satellite instrument observations by the square of the measurement precision and by the product $2\pi^2 R_e^2$ (the effective area of the flattened Earth).

The $\boldsymbol{H}(\varpi)$ vector is constructed from a sequence of terms representing each type of observation on the mission. An entirely different combination of instruments can be investigated by constructing a new transfer function. Position and gravitation gradient observations are represented by the right-hand sides of Eqs. (7) and (9), respectively; Breakwell[1] also supplies expressions for velocity observations and others. A satellite carrying both a three-axis positioning system and a full-tensor gravitation gradiometer (see Secs. III and V) would have an $\boldsymbol{H}(\varpi)$ vector of length 12 and Φ_W^{-1} would be a 12×12 matrix.

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